Quaternion 0x04

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Preview

Preview: Rotating a Point

Goal

$\vec{p} = (x, y, z)$
$\vec{p'} = (x', y', z')$

$$v' = e^{\frac{\pi}{2}n} v e^{-\frac{\pi}{2}n}$$

Method1) Rotation Matrix

Method2) Quaternion

$R^{4\times4}$: matrix for rotation → q: quaternion for rotation

$p = (0, x, y, z)$
$p' = (0, x', y', z')$

$$\begin{bmatrix} \vec{p'} \\ 1 \end{bmatrix} = \mathrm{R}^{4\times4} \begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} \longrightarrow p' = qpq^{-1}$$

Quaternion

What is the Quaternion

***Quaternion**

- Number System extends the complex numbers.
- It were First described by the Irish Mathematician Hamilton in 1843
- 4-dim vector

***Pros**

- Geometric Stability: It prevents gimbal lock
- Computational Efficiency
- Interpolation(SLERP)

William Rowan Hamilton

***Form**

$$q = (a, b, c, d)$$
$$q = a + bi + cj + dk$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Operation of Quaternion

***Quaternion**

$$q_1 = (a, b, c, d) \rightarrow a + bi + cj + dk$$
$$q_2 = (e, f, g, h) \rightarrow e + fi + gj + hk$$

$$i^2 = j^2 = k^2 = ijk = -1$$

***Addition**

$$q_1 + q_2 = (a + e, b + f, c + g, d + h)$$

$$q_1 = (\overbrace{\boxed{a,}}^{\text{Scalar}} \underbrace{\boxed{b, c, d}}_{\text{Vector}})$$

***Multiplication**

$$q_1 q_2 = ?$$

Multiplication of Quaternion

$q_1 q_2 = ?$

$$
\begin{aligned}
&ae + afi + agj + ahk + \\
&bei + bfi^2 + bgij + bhik + \\
&cej + cfji + cgj^2 + chjk + \\
&dek + dfki + dgkj + dhk^2
\end{aligned}
\rightarrow
\begin{aligned}
&ae - bf - cg - dh, \\
&\boxed{\begin{aligned} &be + af - dg + ch, \\ &ce + df + ag - bh, \\ &de - cf + bg + ah \end{aligned}}
\end{aligned}
$$

Imaginary

↓×→	**1**	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	−1	**k**	−**j**
j	**j**	−**k**	−1	**i**
k	**k**	**j**	−**i**	−1

$$i^2 = j^2 = k^2 = ijk = -1$$

$$q_1 = (a, b, c, d) \rightarrow a + bi + cj + dk$$
$$q_2 = (e, f, g, h) \rightarrow e + fi + gj + hk$$

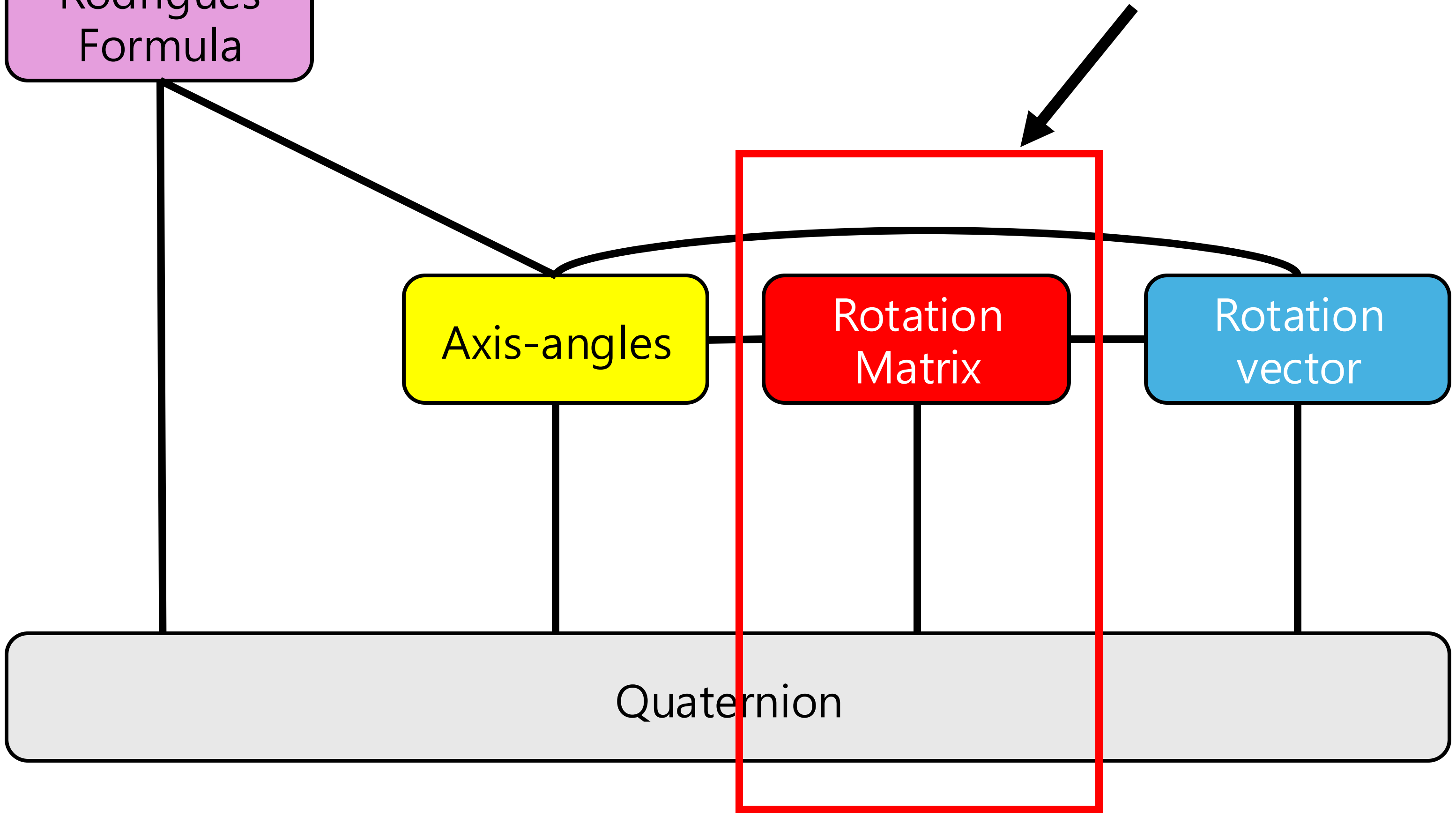
Multiplication of Quaternion

$$q_1 q_2 = (a, b, c, d) \cdot (e, f, g, h) =$$

$$\begin{aligned} &ae - bf - cg - dh, \\ &be + af - dg + ch, \\ &ce + df + ag - bh, \\ &de - cf + bg + ah \end{aligned} \longrightarrow \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$$

Axis-angles

The Relation: Rotation Vector, Rotation Matrix, Quaternion, Rodrigues Formula



Quaternion to Matrix

$$q_1 = (a, b, c, d) \rightarrow a + bi + cj + dk$$

$$q_1 = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

***e.g**

$$(1, 2, 3, 4) = \begin{bmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -4 & 3 \\ 3 & 4 & 1 & -2 \\ 4 & -3 & 2 & 1 \end{bmatrix}$$

Quaternion to Matrix

$$q_1 = (a, b, c, d) \rightarrow a + bi + cj + dk$$

$$\boxed{i^2 = j^2 = k^2 = ijk = -1}$$

$$1 = (1, 0, 0, 0)$$
$$i = (0, 1, 0, 0)$$
$$j = (0, 0, 1, 0)$$
$$k = (0, 0, 0, 1)$$

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Quaternion Representation using Dot product and Cross product

$$q_1 q_2 = (a, b, c, d) \cdot (e, f, g, h) = (a, \vec{x}) \cdot (e, \vec{y})$$

$$\boxed{ae - bf - cg - dh,} \longrightarrow \text{Scalar}$$

$$\boxed{\begin{aligned} &be + af - dg + ch,\\ &ce + df + ag - bh,\\ &de - cf + bg + ah \end{aligned}} \longrightarrow \text{Vector}$$

$$q_1 = (\overbrace{\boxed{a,}}^{\text{Scalar}}\ \underbrace{\boxed{b, c, d}}_{\text{Vector}})$$

***Scalar**

$$ae - (bf + cg + dh) \Rightarrow e\vec{x}$$

Quaternion Representation using Dot product and Cross product

$$q_1q_2 = (a,b,c,d)\cdot(e,f,g,h) = (a,\vec{x})\cdot(e,\vec{y})$$

*Vector

1 2 3

$$\begin{aligned} &be + af - dg + ch, \\ &ce + df + ag - bh, \\ &de - cf + bg + ah \end{aligned}$$

$$1)\, be + ce + de \Rightarrow e\vec{x}$$

$$2)\, af + ag + ah \Rightarrow a\vec{y}$$

$$3)\, \begin{bmatrix} ch - dg \\ df - bh \\ bg - cf \end{bmatrix} = \vec{x}\times\vec{y}$$

$$q_1q_2 \neq q_2q_1$$

*Scalar + Vector

$$q_1q_2 = (a,b,c,d)\cdot(e,f,g,h) = (ae - (bf + cg + dh), \boxed{ae - \vec{x}\cdot\vec{y}, e\vec{x} + a\vec{y} + \vec{x}\times\vec{y})}$$

Special Case) Dot product, Cross product and Quaternion

$$(w_1, \vec{v_1}) \cdot (w_2, \vec{v_2}) = (\boxed{w_1 w_2 - \vec{v_1} \cdot \vec{v_2},} \boxed{w_1 \vec{v_2} + w_2 \vec{v_1} + \vec{v_1} \times \vec{v_2})}$$

$$q_1 = (\overbrace{\boxed{a,}}^{\text{Scalar}} \underbrace{\boxed{b, c, d}}_{\text{Vector}})$$

Scalar Part = 0

$$(0, \vec{v_1}) \cdot (0, \vec{v_2}) = (-\vec{v_1} \cdot \vec{v_2}, \vec{v_1} \times \vec{v_2})$$

$$\text{Let, } v_1 = (0, \vec{v_1}) \text{ and } v_2 = (0, \vec{v_2})$$

$$v_1 v_2 = (-\vec{v_1} \cdot \vec{v_2}, \vec{v_1} \times \vec{v_2})$$

$$v_2 v_1 = (-\vec{v_1} \cdot \vec{v_2}, -\vec{v_1} \times \vec{v_2})$$

***Dot product**

$$v_1 v_2 + v_2 v_1 = ?$$

$$\vec{v_1} \cdot \vec{v_2} = -\frac{1}{2}(v_1 v_2 + v_2 v_1)$$

***Cross product**

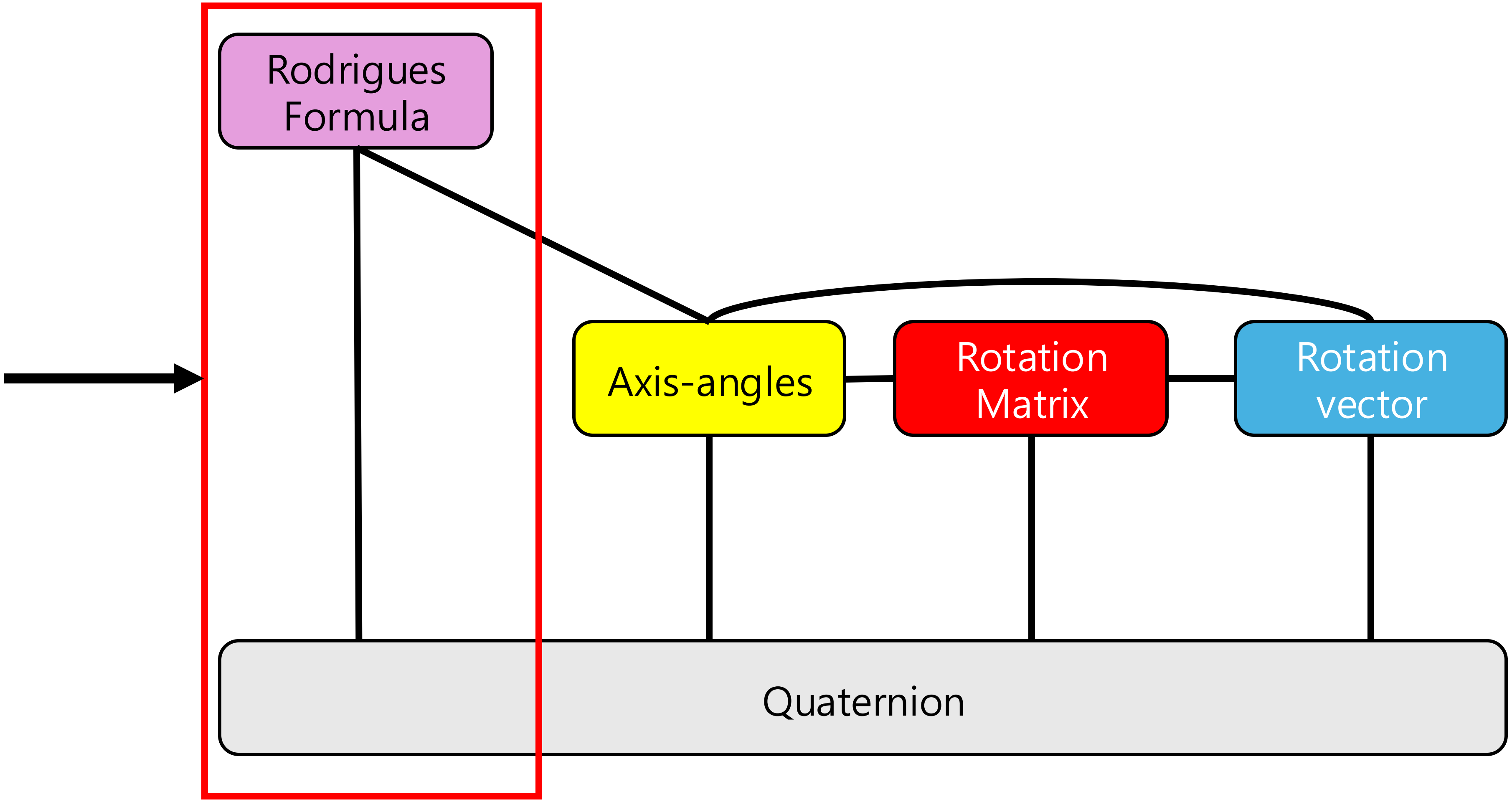
$$v_1 v_2 - v_2 v_1 = ?$$

$$\vec{v_1} \times \vec{v_2} = \frac{1}{2}(v_1 v_2 - v_2 v_1)$$

Quaternion & Rodrigues

Axis-angles

The Relation: Rotation Vector, Rotation Matrix, Quaternion, Rodrigues Formula



Rodrigues Formula

The Rodrigues rotation formula is formula that rotates a vector in three-dimensional space using an arbitrary axis and angle of rotation.

- It allows vectors to be rotated without directly computing rotation matrices.
- high computational efficiency
- computer graphics, robotics, SLAM(Simultaneous Localization And Mapping)

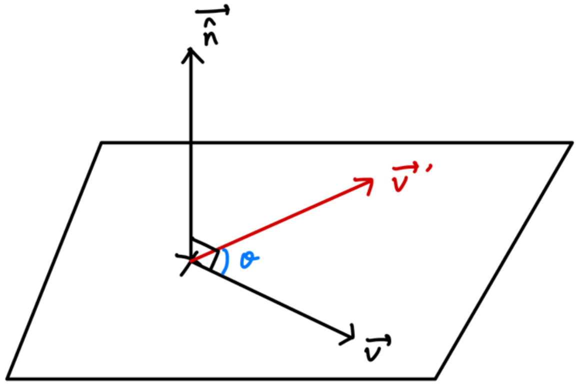
$$\mathbf{v}_{\text{rot}} = \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}\,(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)$$

$$\begin{array}{l} \text{v:}\ \ \text{A Vector} \\ \text{k:}\ \ \text{Rotation-Axis} \\ \theta : angle \end{array} \longrightarrow \mathbf{v}_{\text{rot}}$$

Rodrigues Formula

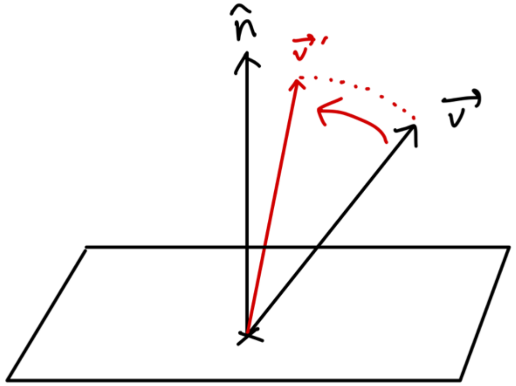
3D rotation

*Special Case



$$\vec{v}' = \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

*General Case



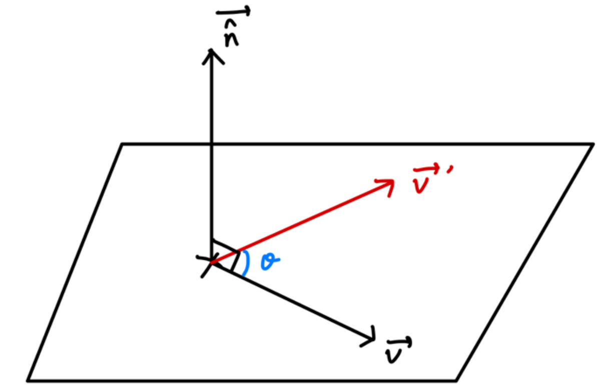
$$\vec{v}' = (1 - \cos\theta)(\vec{v} \cdot \hat{n})\hat{n} + \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

Quaternion and Rodrigues: Special Case

***Form**

$$v = (0, \vec{v})$$
$$v' = (0, \vec{v'})$$
$$n = (0, \hat{n})$$

$$nv = (-\hat{n}\cdot\vec{v}, \hat{n}\times\vec{v}) = (0, \hat{n}\times\vec{v})$$
$$nv = \hat{n}\times\vec{v}$$



***Vector to Quaternion**

$$v' = \cos\theta\cdot v + \sin\theta\cdot(nv)$$
$$v' = (\cos\theta + \sin\theta\cdot n)v$$
$$v' = e^{\theta n}v$$

$$\vec{v'} = \cos\theta\cdot\vec{v} + \sin\theta\cdot\underline{(\hat{n}\times\vec{v})}$$

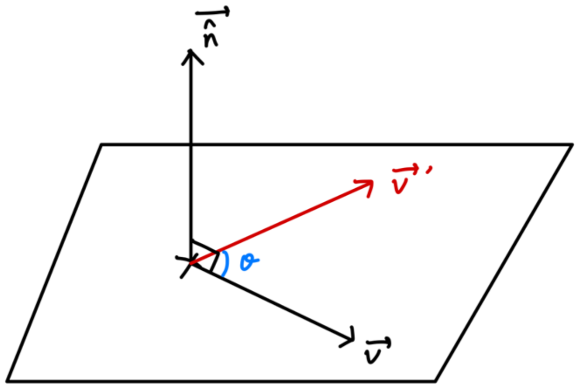
$$\boxed{\cos\theta\cdot\vec{v} + \sin\theta\cdot(\hat{n}\times\vec{v}) \longrightarrow e^{\theta n}v}$$

Vector → Quaternion

Quaternion and Rodrigues: Special Case

***Special Case**

$$\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow e^{\theta n} v$$



$$\vec{v'} = \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$

***e.g**

***input**

$$\vec{v} = (0,0,1)$$
$$\hat{n} = (0,1,0)$$
$$\theta = \frac{\pi}{2}$$

$\longrightarrow$

***input**

$$v = k$$
$$n = j$$

***output**

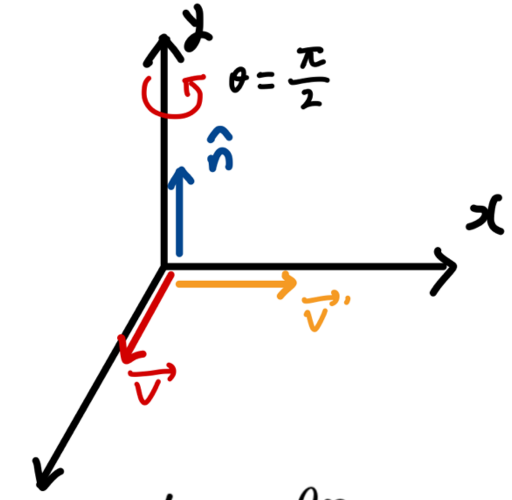
$$\vec{v'} = (?,?,?)$$

***output**

$$v' = ?$$

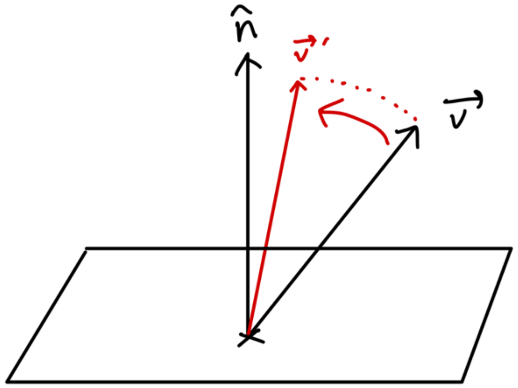
***Equation**

$$v' = e^{\theta n} v \qquad v' = e^{\frac{\pi}{2} j} k \longrightarrow v' = jk \qquad v' = i \longrightarrow v' = (1,0,0)$$



$$v' = e^{\theta n} v$$

Quaternion and Rodrigues: General Case



***Form**

$$v = (0, \vec{v})$$
$$v' = (0, \vec{v'})$$
$$n = (0, \hat{n})$$

$$v_{\parallel} = (0, \vec{v_{\parallel}})$$
$$v_{\perp} = (0, \vec{v_{\perp}})$$
$$v'_{\perp} = (0, \vec{v'_{\perp}})$$

***Vector to Quaternion**

$$v' = \cos\theta \cdot v + \sin\theta \cdot (nv)$$
$$v' = (\cos\theta + \sin\theta \cdot n)v$$
$$v' = e^{\theta n} v$$

$$\vec{v'} = (1 - \cos\theta)(\vec{v} \cdot \hat{n})\hat{n} + \cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v})$$
$$v' = e^{\frac{\pi}{2}n} v e^{-\frac{\pi}{2}n}$$

$$\boxed{\cos\theta \cdot \vec{v} + \sin\theta \cdot (\hat{n} \times \vec{v}) \longrightarrow e^{\frac{\pi}{2}n} v e^{-\frac{\pi}{2}n}}$$

Q&A